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Matrix Formulation for Minimum Response of Undamped Structures

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A useful method for the determination of frequencies that correspond to points of local minima in the frequency response of finite element structure is presented. The method is based on the application of linear matrix algebra methods in conjunction with the fundamental definitions for existence of local minima. The procedure was initially developed for determination of antiresonant frequencies, and is now extended for application to minimum response. The derivations of the procedures for antiresonance and minimum response are both included in this paper. A significant advantage of this method is that it allows for the formulation of sensitivity relationships similar to what is commonly used for resonant frequency. Thus, there is potential for expanded use of minimum response information for structural optimization, damage detection, and other related tasks. In addition, generality in the formulation is retained such that the response may be defined as any linear combination of the coordinate responses, and any arbitrary forcing function may be considered. A numerical example, with a simple one-dimensional spring-mass system, is presented to provide for illustration and validation of the approach.

Nomenclature

$\boldsymbol{A}$	=	system matrix for minimum response formulation
$\boldsymbol{B}$	=	composite mass matrix for minimum response formulation
$\boldsymbol{C}$	=	response operator matrix
$\boldsymbol{e}_i$	=	unit-length vector in direction of coordinate i
$\boldsymbol{F}$	=	force amplitude vector
$\boldsymbol{G}$	=	transformation matrix derived from response definition
$\boldsymbol{K}$	=	analytical stiffness matrix
L	=	number of modes included in modal solution
$\boldsymbol{M}$	=	analytical mass matrix
P_r	=	residue, or response contribution, of rth mode
Resp	=	generalized response variable
$\boldsymbol{T}$	=	transformation matrix derived from input force pattern
$\boldsymbol{U}$	=	frequency-domain displacement response amplitude vector
$\boldsymbol{u}$	=	time-domain displacement response vector
r	=	mode index
x	=	design parameter
α_{ij}	=	receptance frequency response function for ith response coordinate and jth input force coordinate
λ_m	=	minimum response point eigenvalue
μ	=	antiresonant frequency eigenvalue
ϕ_{ir}	=	mode shape component, ith coordinate of the rth mode
ω_r	=	resonant frequency for rth mode
Ω	=	excitation frequency
Ω_a	=	antiresonant frequency
Ω_m	=	minimum response point frequency

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Introduction

RESONANT frequencies and mode shapes inherently define the fundamental dynamic response characteristics of any linear undamped system. For that reason, much research has been devoted to the development of methods for identification of modal parameters and to development of applications that make use of modal parameters. The validity of finite element analysis results requires accurate representation of the system mass and stiffness distributions in the finite element model. Modal tests are performed for measurement of the modal properties of the actual structure, and the resulting data is then used to update and validate the finite element model [1]. Similarly, many approaches have been developed to use measured changes in modal properties for assessment of the health of a system, to detect and identify changes in the system caused by damage [2]. In addition, because excitation frequencies at, or near, natural frequencies of the structure can result in significant response amplifications due to resonance, then it is often necessary to perform structural optimization analyses to determine appropriate design parameters so that the modal properties of the structure can be modified appropriately to reduce the occurrence of damaging resonance conditions [3].

In comparison to resonant response, there has been very little attention in the literature directed toward use of antiresonance, and even less attention has been given to the characteristics associated with local minima of response functions. The presence of local response minima in frequency response functions has traditionally held only theoretical interest [4]. The matrix formulation presented in this paper provides a very interesting approach in that it is possible to formulate an eigenequation that yields eigenvalues corresponding to frequencies of minimum response. Not only does this method provide a feasible approach for the analytical determination of minimum response point frequencies, but it also provides a practical basis for the conduct of sensitivity analyses similar to what is commonly done with natural frequency. Because the frequency response of a system is dependent on the modal properties, then the frequency-dependent characteristics of the response, including the locations of antiresonances and response minima, will be implicitly dependent on the modal properties. The interest here is that minimum response information implicitly carries information from all modes of the system. Consequently, any changes to the system that result in changes to the modes will also result in changes to the minimum response points. Therefore, it is hypothesized that many of the methods currently developed to use modal information for such applications as design optimization or damage assessment could be extended to use minimum response information.

A brief review of minimum response is presented, followed by the derivation of the matrix formulations for antiresonance and minimum response points. Also, the sensitivity relationships that were developed from the formulations are included as well. Finally, a numerical example is presented to illustrate use of the approach, and provide for numerical validation.

Description of Minimum Response

Consider a linear structure that is subjected to sinusoidal excitation forces. The structure will exhibit a steady-state response at the excitation frequency, and the point-to-point variations in the amplitudes and directions of the response will reflect an operating deflection shape. The response level at any given location is dependent on the modal properties of the structure and the particular loading pattern. When the excitation frequency coincides with a natural frequency, then the operating deflection pattern reflects the specific pattern of the mode shape for that frequency, and response amplifications occur due to resonance.

It is expected that the response amplitudes will lower as the excitation frequency is moved away from the natural frequency and then increase as the input frequency approaches the next resonant frequency. Thus, somewhere in the frequency span between two separated resonant frequencies, there must be a frequency for which the response amplitude at a given location will drop to some minimum value, or disappear altogether. Also, because the operating deflection pattern varies with excitation frequency, then a frequency that yields a minimal response at one location will not likely correspond to a minimal response at another location. It is these characteristic frequencies that are of concern in this paper.

The frequency response function (FRF) is a frequency-domain function that describes the amplitude and phase relationship between a response (output) and force (input), as a function of excitation frequency. Graphical representation of this function allows for visualization of the steady-state response characteristics of a structure at a given location. Resonant frequencies are easily recognized as peaks in the FRF. Similarly, those frequencies that exhibit minimal response levels are easily recognized. An example FRF, with these special response points indicated, is given in Fig. 1.

Among those frequencies that produce locally minimal response levels, there are some that theoretically yield no response at a given location. These points represent peaks in an inverse FRF relationship and, for that reason, are labeled as *antiresonant* frequencies. There is a clear distinction, however, between these antiresonant frequencies and those that are labeled herein as *local response minima*, or *minimum response points*, as indicated in Fig. 1. The primary distinction is that, whereas antiresonant points correspond to zero response, minimum response points correspond to a zero slope in the frequency response curve. Also note that, unlike antiresonant response, in the vicinity of a local minimum response point the response level is relatively stable to changes in frequency.

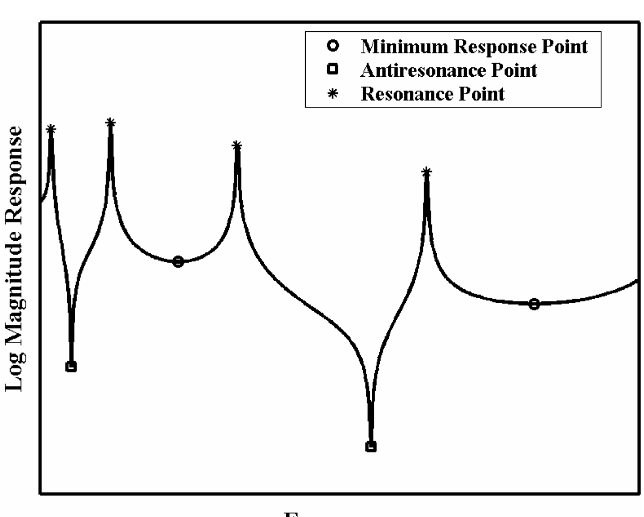

Fig. 1 Illustration of characteristic points in the FRF.

One property that is shared between antiresonant responses and local response minima is that the frequencies of both are nonglobal. That is, each and every response location will exhibit a unique set of local minima and antiresonant frequencies. This is unlike the global nature of natural frequencies. Resonant frequencies are system characteristics and, as such, are evident in varying degrees in the responses at all locations. Measurement of response from one location may reveal all of the natural frequencies within the measured frequency range, but then other locations will contribute no new information. This uniqueness offered by antiresonant frequency information has been shown to provide localization of damage in symmetrical structures that was not possible with resonant frequency data alone [5]. It is expected that the same benefit could be derived from minimum response point information as well.

Mathematical Examination of Minimum Response

In general, minimum response information can be extracted from any frequency-domain response function. However, for purpose of clarity, this discussion is based on the *receptance* form of the frequency response function. The receptance defines the frequency response relationship between a displacement response and an applied force. The mathematical representation, based on the principal of mode superposition, for the receptance (α_{ij}) between a response at coordinate i and a force at coordinate j, of a linear undamped multiple degree-of-freedom (DOF) system is given by Eq. (1), with L mass-normalized modes included [6].

$$\alpha_{ij}(\Omega) = \sum_{r=1}^{L} \frac{\phi_{ir}\phi_{jr}}{\omega_r^2 - \Omega^2} \tag{1}$$

Based on Eq. (1), a typical response can be further generalized as follows:

$$\text{Resp}\,(\Omega) = \sum_{r=1}^{L} \frac{P_r}{\omega_r^2 - \Omega^2} \tag{2}$$

Now, computing the first and second derivatives of the response with respect to the square of the excitation frequency yields the following:

$$\text{Res p}' = \frac{\text{dResp}}{\text{d}(\Omega^2)} = \sum_{r=1}^{L} \frac{P_r}{\left(\omega_r^2 - \Omega^2\right)^2} \tag{3}$$

$$\text{Res p}'' = \frac{\text{d}^2\text{Resp}}{\text{d}(\Omega^2)^2} = \sum_{r=1}^{L} \frac{2P_r}{\left(\omega_r^2 - \Omega^2\right)^3} \tag{4}$$

Because antiresonant frequencies correspond to responses of zero amplitude, then they are found where $\text{Resp}(\Omega_a) = 0$. Thus, the antiresonant frequencies are found as the roots of Eq. (5) as follows:

$$\text{Resp}\,(\Omega_a) = \frac{P_1}{\omega_1^2 - \Omega_a^2} + \frac{P_2}{\omega_2^2 - \Omega_a^2} + \cdots + \frac{P_L}{\omega_L^2 - \Omega_a^2} = 0 \tag{5}$$

Likewise, the frequencies that correspond to local response minima are found where the rate of change of the response with respect to frequency is zero. That is, possible minimum response frequencies correspond to the roots of the following equation:

$$\text{Resp}'(\Omega_m) = \frac{P_1}{(\omega_1^2 - \Omega_m^2)^2} + \frac{P_2}{(\omega_2^2 - \Omega_m^2)^2} + \cdots + \frac{P_L}{(\omega_L^2 - \Omega_m^2)^2} = 0 \tag{6}$$

Once a candidate minimum response frequency is identified from Eq. (6), then it is necessary to perform a test to determine whether or not the frequency corresponds to a point of minimum or maximum response. If either of the following two conditions are true, then Ω_m corresponds to a minimum response frequency:

$$\begin{aligned} &1)\ \text{Resp}(\Omega_m) > 0 \quad \text{and} \quad \text{Resp}''(\Omega_m) > 0 \\ &2)\ \text{Resp}(\Omega_m) < 0 \quad \text{and} \quad \text{Resp}''(\Omega_m) < 0 \end{aligned} \tag{7}$$

However, only for very simple problems can direct solutions of Eqs. (5) and (6) be performed. For larger problems, it is not possible to perform direct solutions, and even numerical solutions seem unrealistic if a large number of modes are included in the process. An approximation for an antiresonance, or minimum response frequency, can be performed with the assumption that the response over the frequency span between two resonances is dominated by only those two modes. Assume for the moment that the first two modes completely dominate the response over the frequency range, $\omega_1 < \Omega < \omega_2$. Then an antiresonant frequency exists at $\Omega = \Omega_a$, if

$$\frac{P_1}{\omega_1^2 - \Omega_a^2} + \frac{P_2}{\omega_2^2 - \Omega_a^2} = 0 \tag{8}$$

Likewise, a minimum response point can be found at $\Omega = \Omega_m$, if

$$\frac{P_1}{\left(\omega_1^2 - \Omega_m^2\right)^2} + \frac{P_2}{\left(\omega_2^2 - \Omega_m^2\right)^2} = 0 \tag{9}$$

Note that Eq. (8) can only be satisfied for an antiresonant frequency if P_1 and P_2 are of the same sign or $P_1 \cdot P_2 > 0$. Likewise, if all P_r are of the same sign, such that $(P_r \cdot P_{r+1}) > 0$ for all indices r between 1 and $(N-1)$, then there will be $N-1$ antiresonant frequencies and no minimum response point frequencies present for this condition. Note also that Eq. (9) can only be satisfied for a minimum response frequency on the condition that $P_1 \cdot P_2 < 0$. The existence criteria can be summarized as follows:

1) $P_r \cdot P_{r+1} > 0 \Rightarrow \omega_r < \Omega_a < \omega_{r+1}$
(no minimum response frequency)

2) $P_r \cdot P_{r+1} < 0 \Rightarrow \omega_r < \Omega_m < \omega_{r+1}$
(no antiresonant frequency) (10)

3) P_r is same sign for all $r = 1, \ldots, N \Rightarrow (N-1)$ antiresonances
(no minimum response frequencies)

4) P_r alternating sign for $r = 1, \ldots, N$
$\Rightarrow (N-1)$ minimum response frequencies
(no antiresonances)

These statements could be used as test conditions in a search process. If the condition is satisfied, then a solution could be determined for the minimum response frequency located between the two modes. If the condition is not met, then the test condition could be applied to the next frequency span in the search process. This is still an undesirable approach, because this formulation does not allow a direct means for determination of the sensitivity of minimum response frequencies to changes in the physical parameters of the system. An improved formulation for the determination of minimum response frequencies, that does provide an effective means for sensitivity analysis techniques, is presented in the following section.

Matrix Formulation for Minimum Response

A matrix formulation for the direct computation of minimum response point frequencies from finite element mass and stiffness distribution matrices is derived here. Given the finite element mass and stiffness matrices, an eigenvalue problem can be defined for a particular response definition and input force pattern, such that the frequencies of minimum response will be contained in the corresponding eigenvalue set. The primary emphasis here is directed toward minimum response. However, a similar matrix formulation was previously developed for determination of antiresonant frequencies [7] and is included herein as well.

The equation of motion for an undamped discrete system under steady-state harmonic excitation, at frequency Ω, is given by $\boldsymbol{M\ddot{u}} + \boldsymbol{Ku} = \boldsymbol{F}e^{i\Omega t}$ [8]. For the steady-state response solution, $\boldsymbol{u} = \boldsymbol{U}e^{i\Omega t}$, the steady-state form of the equation of motion can be represented as

$$(\boldsymbol{K} - \Omega^2\boldsymbol{M})\boldsymbol{U} = \boldsymbol{F} \tag{11}$$

A general response may be defined by any linear combination of the physical coordinates of the model. This representation is expressed as

$$\text{Resp} = \boldsymbol{CU} \tag{12}$$

It is typically desired to find the minimum response points associated with the response at a particular location. In this case, the response of interest will include only the contribution of a single coordinate. The response of the ith DOF, U_i, can be expressed as

$$\text{Resp} = \boldsymbol{e}_i^T\boldsymbol{U}$$

It may also be desired to define the generalized response variable in terms of relative displacements for representation of strain response or spring-force quantities. In this case, a response representing the relative deformation between coordinates i and j, for example, could be expressed as

$$\text{Resp} = (\boldsymbol{e}_i - \boldsymbol{e}_j)^T\boldsymbol{U} \tag{13}$$

Matrix Formulation for Antiresonant Frequency

Before discussion of a formulation for minimum response, a formulation for antiresonant frequency is first presented [7]. For the determination of antiresonant frequencies, the goal is to find Ω_a, such that $\text{Resp}(\Omega_a) = \boldsymbol{CU} = 0$. The components in $\boldsymbol{U}$ are not independent. Thus, $\boldsymbol{U}$ can be redefined as follows:

$$\boldsymbol{U} = \boldsymbol{Gq} \tag{14}$$

so that

$$\text{Resp}\,(\Omega_a) = \boldsymbol{CGq} = 0 \tag{15}$$

For Eq. (15) to be satisfied, $\boldsymbol{G}$ must be defined as the null space of $\boldsymbol{C}$. That is,

$$\boldsymbol{G} \equiv \text{null space of } \boldsymbol{C} \Rightarrow \boldsymbol{CG} = 0 \tag{16}$$

Substituting Eq. (14) into Eq. (11) yields the following:

$$(\boldsymbol{K} - \Omega^2\boldsymbol{M})\boldsymbol{Gq} = \boldsymbol{F} \tag{17}$$

Suppose that another transformation matrix, $\boldsymbol{T}$, which is soon to be defined, is transposed and premultiplied into Eq. (17) as follows:

$$\boldsymbol{T}^T(\boldsymbol{K} - \Omega^2\boldsymbol{M})\boldsymbol{Gq} = \boldsymbol{T}^T\boldsymbol{F} = (\boldsymbol{F}^T\boldsymbol{T})^T \tag{18}$$

Note that if $\boldsymbol{T}$ is explicitly defined as the *null space* of $\boldsymbol{F}^T$, then the right side of Eq. (18) becomes zero. For this definition of $\boldsymbol{T}$, Eq. (18) can then be transformed into the generalized eigenvalue problem shown next. That is, for

$$\boldsymbol{T} \equiv \text{null space of } \boldsymbol{F}^T \tag{19}$$

then

$$(\bar{\boldsymbol{K}} - \Omega^2 \bar{\boldsymbol{M}})\boldsymbol{q} = 0 \tag{20}$$

where

$$\bar{\boldsymbol{K}} = \boldsymbol{T}^T \boldsymbol{K} \boldsymbol{G} \quad \text{and} \quad \bar{\boldsymbol{M}} = \boldsymbol{T}^T \boldsymbol{M} \boldsymbol{G} \tag{21}$$

Because $\bar{\boldsymbol{K}}$ and $\bar{\boldsymbol{M}}$ are generally nonsymmetric matrices, then there will likely be real and complex-quantity eigenvalues. Only the positive–real eigenvalues correspond to the antiresonant frequencies for the particular response and input combination.

Matrix Formulation for Minimum Response Point

The formulation for determination of minimum response frequencies is an extension of the formulation found in [7] and presented in the preceding section for antiresonant frequency. First, it is necessary to compute the derivatives of the response $\boldsymbol{U}$ for determination of minimum response point frequencies. Differentiating both sides of Eq. (11) with respect to Ω^2 yields

$$-\boldsymbol{M}\boldsymbol{U} + (\boldsymbol{K} - \Omega^2 \boldsymbol{M})\boldsymbol{U}' = \boldsymbol{F}' \tag{22}$$

where $\boldsymbol{U}' = \mathrm{d}\boldsymbol{U}/\mathrm{d}(\Omega^2)$, and, $\boldsymbol{F}' = \mathrm{d}\boldsymbol{F}/\mathrm{d}(\Omega^2)$. The matrix equations expressed by Eqs. (11) and (22) can be combined into a single matrix representation as

$$\begin{bmatrix} \boldsymbol{K} - \Omega^2 \boldsymbol{M} & 0 \\ -\boldsymbol{M} & \boldsymbol{K} - \Omega^2 \boldsymbol{M} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{U}' \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F} \\ \boldsymbol{F}' \end{Bmatrix} \tag{23}$$

For any frequency-independent force vector, $\boldsymbol{F}' = 0$. Equation (23) can then be simplified and rearranged as follows:

$$\left(\begin{bmatrix} \boldsymbol{K} & 0 \\ -\boldsymbol{M} & \boldsymbol{K} \end{bmatrix} - \Omega^2 \begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & \boldsymbol{M} \end{bmatrix} \right) \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{U}' \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F} \\ 0 \end{Bmatrix} \tag{24}$$

Recalling Eqs. (12) and (13) for definition of the response from which minimum response point frequencies are desired, the goal here is to find Ω_m such that $\text{Resp}' = \boldsymbol{C}\boldsymbol{U}' = 0$. Thus, the following combined matrix expression can be written

$$\begin{bmatrix} 0 & \boldsymbol{C} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{U}' \end{Bmatrix} = 0 \tag{25}$$

Because the components of $\boldsymbol{U}$ and $\boldsymbol{U}'$ are not independent, then the following coordinate transformation may be applied:

$$\begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{U}' \end{Bmatrix} = \boldsymbol{G}\boldsymbol{q} \tag{26}$$

and, if Eq. (26) is substituted back into Eq. (25), then

$$\begin{bmatrix} 0 & \boldsymbol{C} \end{bmatrix} \boldsymbol{G}\boldsymbol{q} = 0 \tag{27}$$

For Eq. (27) to be satisfied, $\boldsymbol{G}$ must be defined as the null space of the composite matrix:

$$\begin{bmatrix} 0 & \boldsymbol{C} \end{bmatrix}$$

That is,

$$\boldsymbol{G} \;\equiv\; \text{null space of } \left(\begin{bmatrix} 0 & \boldsymbol{C} \end{bmatrix} \right) \Rightarrow \begin{bmatrix} 0 & \boldsymbol{C} \end{bmatrix} \boldsymbol{G} = 0 \tag{28}$$

Equation (26) can then be substituted back into Eq. (24) to yield the following:

$$(\boldsymbol{A} - \Omega^2 \boldsymbol{B})\boldsymbol{G}\boldsymbol{q} = \begin{Bmatrix} \boldsymbol{F} \\ 0 \end{Bmatrix} \tag{29}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{K} & 0 \\ -\boldsymbol{M} & \boldsymbol{K} \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & \boldsymbol{M} \end{bmatrix} \tag{30}$$

In similar fashion to what was done with regard to antiresonance, $\boldsymbol{T}$, to be defined later, can be transposed and premultiplied into Eq. (29) as follows:

$$\boldsymbol{T}^T (\boldsymbol{A} - \Omega^2 \boldsymbol{B})\boldsymbol{G}\boldsymbol{q} = \boldsymbol{T}^T \begin{Bmatrix} \boldsymbol{F} \\ 0 \end{Bmatrix} = \left(\begin{bmatrix} \boldsymbol{F}^T & 0 \end{bmatrix} \boldsymbol{T} \right)^T \tag{31}$$

Note that if $\boldsymbol{T}$ is explicitly defined as the null space of

$$\begin{bmatrix} \boldsymbol{F}^T & 0 \end{bmatrix}$$

such that

$$\begin{bmatrix} \boldsymbol{F}^T & 0 \end{bmatrix} \boldsymbol{T} = 0$$

then Eq. (31) can be reduced into the generalized eigenvalue problem shown next. That is, for

$$\boldsymbol{T} \;\equiv\; \text{null space of } \begin{Bmatrix} \boldsymbol{F} \\ 0 \end{Bmatrix}^T \tag{32}$$

then

$$(\bar{\boldsymbol{A}} - \Omega^2 \bar{\boldsymbol{B}})\boldsymbol{q} = 0 \tag{33}$$

where

$$\bar{\boldsymbol{A}} = \boldsymbol{T}^T \boldsymbol{A} \boldsymbol{G} \quad \text{and} \quad \bar{\boldsymbol{B}} = \boldsymbol{T}^T \boldsymbol{B} \boldsymbol{G} \tag{34}$$

Because $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{B}}$ are generally nonsymmetric matrices, then there will likely be real and complex-quantity eigenvalues. Only the positive–real eigenvalues will be candidate values for the zero-slope minimum response point frequencies for the particular response and input combination. It is necessary to check the second derivative of the response to determine whether or not a candidate eigenvalue corresponds to a point of minimum or maximum response. Computing the second derivative of Eq. (11) with respect to Ω^2 yields

$$(\boldsymbol{K} - \Omega^2 \boldsymbol{M})\boldsymbol{U}'' - 2\boldsymbol{M}\boldsymbol{U}' = \boldsymbol{F}'' \tag{35}$$

Once $\boldsymbol{U}''$ is known, then the second derivative of the response can be determined as

$$\text{Resp}'' = \boldsymbol{C}\boldsymbol{U}'' \tag{36}$$

Recalling the minimum response point test conditions given by Eq. (7) for the candidate eigenvalue $\bar{\Omega}^2$:

1) If $\text{Resp}(\bar{\Omega}) > 0$, then $\bar{\Omega} = \Omega_m$ if $\text{Resp}''(\bar{\Omega}) > 0$ or

2) If $\text{Resp}(\bar{\Omega}) < 0$, then $\bar{\Omega} = \Omega_m$ if $\text{Resp}''(\bar{\Omega}) < 0$

Sensitivity of Minimum Response

For a variety of applications, including design optimization, model update tasks, and damage detection, it is typically necessary to quantify changes in the natural frequencies and mode shapes of the structure that result from changes in various design parameters. These design parameters may include material properties, geometrical dimensions, etc. An efficient process for evaluation of these changes is through reanalysis techniques that are based on the use of sensitivity relationships obtained directly from the generalized eigenvalue problem derived from the motion equations for the system. Given some design parameter x, the rate of change of the mass and stiffness matrix components with respect to this parameter can be determined. Then, the sensitivity of each resonant frequency can be easily estimated [9]. Because it has been shown that an eigenvalue problem definition for minimum response is available, then it is also theoretically possible to compute the rate of change of minimum response point frequencies to any design parameters.

Sensitivity of Antiresonant Frequency

The sensitivity relationships contained in [7] for antiresonance are included here for completeness with respect to the discussion of minimum response. Recall the eigenequation previously derived for antiresonant response points, defined by Eqs. (20) and (21). This relationship is slightly rearranged and expressed in terms of the eigenvalue, where $\mu = \Omega_a^2$ as follows:

$$\bar{\boldsymbol{K}}\boldsymbol{q} = \mu\bar{\boldsymbol{M}}\boldsymbol{q} \tag{37}$$

The partial derivative of both sides of Eq. (37) with respect to some design parameter, denoted as x, yields the following:

$$(\bar{\boldsymbol{K}} - \mu\bar{\boldsymbol{M}})\frac{\partial \boldsymbol{q}}{\partial x} - \frac{\partial \mu}{\partial x}\bar{\boldsymbol{M}}\boldsymbol{q} = -\left(\frac{\partial \bar{\boldsymbol{K}}}{\partial x} - \mu\frac{\partial \bar{\boldsymbol{M}}}{\partial x}\right)\boldsymbol{q} \tag{38}$$

The matrices $\bar{\boldsymbol{K}}$ and $\bar{\boldsymbol{M}}$ will generally be nonsymmetric. Thus, consider also, the corresponding adjoint eigenvalue equation associated with these nonsymmetric matrices. That is, the transposed forms of $\bar{\boldsymbol{K}}$ and $\bar{\boldsymbol{M}}$ will also form an eigenequation that yields the same set of eigenvalues and a complimentary set of eigenvectors. This adjoint problem is expressed as

$$\bar{\boldsymbol{K}}^T\boldsymbol{p} = \mu\boldsymbol{M}^T\boldsymbol{p} \tag{39}$$

where $\boldsymbol{p}$ is defined as an eigenvector of the adjoint problem. The physical significance of the adjoint problem is easily understood upon consideration of the property of reciprocity for a linear structure. That is, if the response and force coordinates are swapped, then the same transfer function will be obtained, because the dynamic stiffness matrix for a linear structure is symmetric and, consequently, the same antiresonant frequencies will be present. However, the spatial variation of the responses at the antiresonant frequencies will differ between the swapped and unswapped conditions, consistent with the difference in eigenvectors between Eqs. (37) and (39). Now, if Eq. (38) is transposed, and postmultiplied by $\boldsymbol{p}$, then the following result is obtained:

$$\frac{\partial \boldsymbol{q}^T}{\partial x}(\bar{\boldsymbol{K}}^T - \mu\bar{\boldsymbol{M}}^T)\boldsymbol{p} - \frac{\partial \mu}{\partial x}\boldsymbol{q}^T\bar{\boldsymbol{M}}^T\boldsymbol{p} = -\boldsymbol{q}^T\left(\frac{\partial \bar{\boldsymbol{K}}}{\partial x} - \mu\frac{\partial \bar{\boldsymbol{M}}^T}{\partial x}\right)\boldsymbol{p} \tag{40}$$

The first term in Eq. (40) can be eliminated in accordance with Eq. (39) and, after some rearrangement, the eigenvalue sensitivity can be expressed as

$$\frac{\partial \mu}{\partial x} = \frac{\boldsymbol{q}^T[(\partial\bar{\boldsymbol{K}}^T/\partial x) - \mu(\partial\bar{\boldsymbol{M}}^T/\partial x)]\boldsymbol{p}}{\boldsymbol{q}^T\bar{\boldsymbol{M}}^T\boldsymbol{p}} \tag{41}$$

The rate of change of the antiresonant frequency is related to the eigenvalue sensitivity by

$$\frac{\partial \Omega_a}{\partial x} = \frac{1}{2\Omega_a}\frac{\partial \mu}{\partial x} \tag{42}$$

Sensitivity of Minimum Response Frequency

Recall the eigenequation previously derived for minimum response points, defined by Eqs. (33) and (34), and slightly rearranged as follows:

$$\bar{\boldsymbol{A}}\boldsymbol{q} = \lambda_m\bar{\boldsymbol{B}}\boldsymbol{q} \tag{43}$$

The partial derivative of both sides of Eq. (43) with respect to some design parameter, denoted as x, can be expressed as

$$(\bar{\boldsymbol{A}} - \lambda_m\bar{\boldsymbol{B}})\frac{\partial \boldsymbol{q}}{\partial x} - \frac{\partial \lambda_m}{\partial x}\bar{\boldsymbol{B}}\boldsymbol{q} = -\left(\frac{\partial \bar{\boldsymbol{A}}}{\partial x} - \lambda_m\frac{\partial \bar{\boldsymbol{B}}}{\partial x}\right)\boldsymbol{q} \tag{44}$$

The matrices $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{B}}$ will generally be nonsymmetric. Thus, just as in the formulation derived for antiresonance, there will be a corresponding adjoint eigenvalue equation for the nonsymmetric matrices $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{B}}$ for minimum response. That is, the transposed forms of $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{B}}$ will also form an eigenequation that yields the same set of eigenvalues and a complimentary set of eigenvectors. This adjoint problem is expressed as

$$\bar{\boldsymbol{A}}^T\boldsymbol{p} = \lambda_m\bar{\boldsymbol{B}}^T\boldsymbol{p} \tag{45}$$

where $\boldsymbol{p}$ is defined as an eigenvector of the adjoint problem. If Eq. (44) is transposed, and postmultiplied by $\boldsymbol{p}$, then the following result is obtained:

$$\frac{\partial \boldsymbol{q}^T}{\partial x}(\bar{\boldsymbol{A}}^T - \lambda_m\bar{\boldsymbol{B}}^T)\boldsymbol{p} - \frac{\partial \lambda_m}{\partial x}\boldsymbol{q}^T\bar{\boldsymbol{B}}^T\boldsymbol{p} = -\boldsymbol{q}^T\left(\frac{\partial \bar{\boldsymbol{A}}^T}{\partial x} - \lambda_m\frac{\partial \bar{\boldsymbol{B}}^T}{\partial x}\right)\boldsymbol{p} \tag{46}$$

The first term in Eq. (46) can be eliminated in accordance with Eq. (45) and, after some rearrangement, the eigenvalue sensitivity can be expressed as

$$\frac{\partial \lambda_m}{\partial x} = \frac{\boldsymbol{q}^T[(\partial\bar{\boldsymbol{A}}^T/\partial x) - \lambda_m(\partial\bar{\boldsymbol{B}}^T/\partial x)]\boldsymbol{p}}{\boldsymbol{q}^T\bar{\boldsymbol{B}}^T\boldsymbol{p}} \tag{47}$$

Equation (47) represents the general sensitivity relationship for the minimum response eigenvalue. For many applications, a simplified form of Eq. (47) may be possible. If all elements of the mass matrix are independent of the parameter x, such that $\partial\boldsymbol{M}/\partial x = 0 \Rightarrow \partial\bar{\boldsymbol{B}}/\partial x = 0$, and if the associated eigenvectors are normalized such that $\boldsymbol{q}^T\bar{\boldsymbol{B}}^T\boldsymbol{p} = 1$, then Eq. (47) can be simplified to

$$\frac{\partial \lambda_m}{\partial x} = \boldsymbol{q}^T\frac{\partial \bar{\boldsymbol{A}}^T}{\partial x}\boldsymbol{p} \;\Rightarrow\; \frac{\partial \lambda_m}{\partial x} = \boldsymbol{q}^T\boldsymbol{G}^T\begin{bmatrix}\partial\boldsymbol{K}/\partial x & 0\\ 0 & \partial\boldsymbol{K}/\partial x\end{bmatrix}\boldsymbol{T}\boldsymbol{p} \tag{48}$$

Generalized forms of the eigenvectors $\boldsymbol{q}$ and $\boldsymbol{p}$ can be defined by $\boldsymbol{\psi}$ and $\boldsymbol{\beta}$ as follows:

$$\boldsymbol{\psi} = \boldsymbol{G}\boldsymbol{q}, \qquad \boldsymbol{\beta} = \boldsymbol{T}\boldsymbol{p} \tag{49}$$

Finally, the generalized eigenvectors are then partitioned to facilitate expansion of Eq. (48) into a more familiar form. The minimum response eigenvalue sensitivity, with an independent mass matrix, can then be represented as

$$\frac{\partial \lambda_m}{\partial x} = \psi_u^T\frac{\partial \boldsymbol{K}}{\partial x}\beta_u + \psi_l^T\frac{\partial \boldsymbol{K}}{\partial x}\beta_l \tag{50}$$

where

$$\boldsymbol{\psi} = \begin{Bmatrix}\boldsymbol{\psi}_u\\ \boldsymbol{\psi}_l\end{Bmatrix}, \qquad \boldsymbol{\beta} = \begin{Bmatrix}\boldsymbol{\beta}_u\\ \boldsymbol{\beta}_l\end{Bmatrix} \tag{51}$$

The form given by Eq. (50) is very similar to the reduced form that relates the sensitivity of natural frequency eigenvalues to derivatives of the stiffness matrix for the case of an independent mass matrix [8].

Numerical Example

The 3-DOF undamped discrete spring-mass system, illustrated in Fig. 2, was chosen to provide for a simple numerical demonstration of the matrix formulations for minimum response.

Considering single-point excitation applied at DOF 3, the desire in this example is to determine the location of the minimum response frequencies for the DOF 2 response (U_2). The mass and stiffness

Data: $K_1 = 100$ N/m, $M_1 = 3$ kg, $F_1 = 0$
$K_2 = 100$ N/m, $M_2 = 2$ kg, $F_2 = 0$
$K_3 = 100$ N/m, $M_3 = 1$ kg, $F_3 = 1$ N

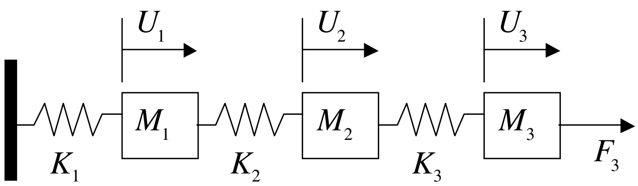

Fig. 2 Example 3-DOF system.

matrices of this system are easily constructed and are given along with the input force vector as follows:

$$\boldsymbol{M} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{kg}$$

$$\boldsymbol{K} = \begin{bmatrix} 200 & -100 & 0 \\ -100 & 200 & -100 \\ 0 & -100 & 100 \end{bmatrix} \text{N/m}, \qquad \boldsymbol{F} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \text{N}$$

First, the direct solution of Eq. (11) for the response $\boldsymbol{U}$ is examined. It can be shown that the response solution, $\text{Resp} = U_2$, and the corresponding first and second derivatives, as defined by Eqs. (3) and (4), are as follows:

$$\text{Resp} = \frac{50(3\Omega^2 - 200)}{(3\Omega^6 - 800\Omega^4 + 50{,}000\Omega^2 - 500{,}000)} \tag{52}$$

$$\text{Resp}' = \frac{-100(9\Omega^6 - 2100\Omega^4 + 160{,}000\Omega^2 - 4{,}250{,}000)}{(3\Omega^6 - 800\Omega^4 + 50{,}000\Omega^2 - 500{,}000)^2} \tag{53}$$

$$\text{Resp}'' = \frac{300(27\Omega^{10} - 10{,}800\Omega^8 + 1{,}770{,}000\Omega^6 - 149 \times 10^6\Omega^4 + 65 \times 10^8\Omega^2 - 115 \times 10^9)}{(3\Omega^6 - 800\Omega^4 + 50{,}000\Omega^2 - 500{,}000)^3} \tag{54}$$

Antiresonant frequencies must satisfy the condition that $\text{Resp}(\Omega_m^2) = 0$, and from Eq. (52) it is easily recognized that $\Omega_a = \sqrt{200/3}$ rad/s. The minimum response frequencies corresponding to $\text{Resp}'(\Omega_m^2) = 0$, from Eq. (53), are more difficult to determine. Application of a root solver will result in the following frequency values that satisfy the zero-slope condition: $(\Omega_1)^2 = 112.96$, $(\Omega_2)^2 = 60.19 + 23.62i$, and $(\Omega_3)^2 = 60.19 - 23.62i$, where here $i = \sqrt{-1}$. There is only one real-valued root and thus there is only one candidate minimum response point frequency, located at $\Omega_m = 10.628$ rad/s. At this frequency, $\text{Resp} = -.0094$ and $\text{Resp}'' = -5.56 \times 10^{-6}$. Because both are negative-valued, then in accordance with Eq. (7), the associated frequency value is identified as a minimum response point frequency.

For comparison, the matrix formulation method is demonstrated next. Because the response of interest is defined by U_2 alone, then

$$\text{Resp} = U_2 \Rightarrow \boldsymbol{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

For evaluation of antiresonant frequencies, the associated transformation matrices, $\boldsymbol{G}$ and $\boldsymbol{T}$, as defined by Eqs. (16) and (19), respectively, are evaluated as

$$\boldsymbol{G} = \text{null}(\boldsymbol{C}) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{T} = \text{null}(\boldsymbol{F}^T) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The corresponding eigenvalue problem, as defined by Eq. (20) and (21), becomes

$$\left(\begin{bmatrix} 100 & -100 \\ 200 & 0 \end{bmatrix} - \Omega^2 \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}\right)\boldsymbol{q} = 0$$

from which there is only one real eigenvalue, such that $\Omega_a = 8.165 \text{ rad/s} = \sqrt{200/3}$ (same as the direct result).

For evaluation of minimum response point frequencies, the associated transformation matrices $\boldsymbol{G}$ and $\boldsymbol{T}$, as defined by Eqs. (28) and (32), respectively, are evaluated as

$$\boldsymbol{G} = \text{null}\left(\begin{bmatrix} 0 & \boldsymbol{C} \end{bmatrix}\right) = \text{null}\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}\right)$$

and

$$\boldsymbol{T} = \text{null}\left(\begin{bmatrix} \boldsymbol{F}^T & 0 \end{bmatrix}\right) = \text{null}\left(\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$\Rightarrow \boldsymbol{G} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{T} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The corresponding eigenvalue problem, as defined by Eqs. (33) and (34), becomes

$$\left(\begin{bmatrix} 200 & -100 & 0 & 100 & 0 \\ 100 & 0 & 0 & 200 & 0 \\ 0 & 0 & 200 & 3 & 0 \\ -2 & 0 & -100 & 0 & -100 \\ 0 & -1 & 0 & 0 & 100 \end{bmatrix} - \Omega^2 \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right)\boldsymbol{q} = 0$$

from which there are three finite-valued eigenvalues, with one being positive–real and the other two belonging to a complex–conjugate pair, precisely as was determined from the direct method. That is, these eigenvalues were determined to be $(\Omega_1)^2 = 112.96$, $(\Omega_2)^2 = 60.19 + 23.62i$, and $(\Omega_3)^2 = 60.19 - 23.62i$. Thus, there is only one candidate minimum response point frequency, and it is confirmed through comparison of the response and second derivative of the response that the minimum response point frequency is

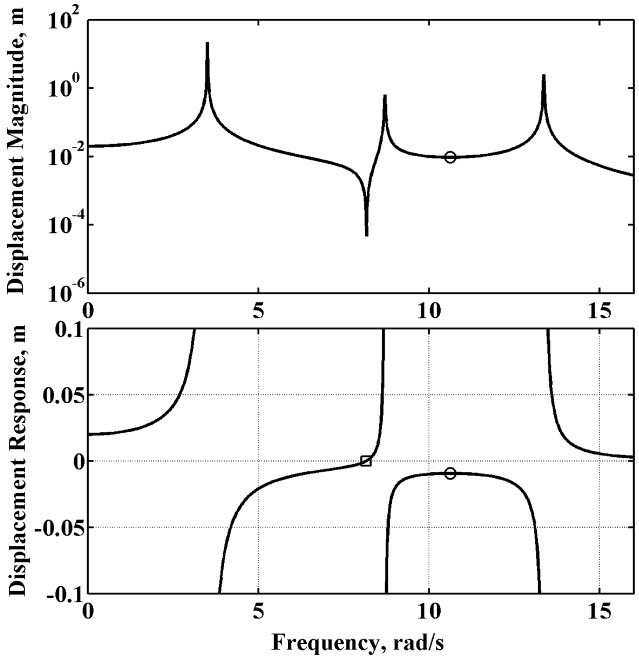

Fig. 3 Illustration of matrix formulation results.

$\Omega_m = 10.628$ rad/s. For illustration, these points are indicated on the frequency response plot for U_2 shown in Fig. 3.

A modal transformation can be used to obtain an efficient response solution for the system subjected to the given steady-state force at DOF 3. In this way, the response solution is formulated in terms of mass-normalized mode shapes and natural frequencies, as follows:

$$U_i = \sum_{r=1}^{3} \phi_{ir}\phi_{3r} / \left(\omega_r^2 - \Omega^2\right) \tag{55}$$

The natural frequencies were found at $\omega_1 = 3.5092$ rad/s, $\omega_2 = 8.7065$ rad/s, and $\omega_3 = 13.3623$ rad/s, and the mass-normalized mode shapes are listed as follows:

$$\phi_1 = \begin{Bmatrix} -0.29141 \\ -0.47517 \\ -0.54190 \end{Bmatrix}, \qquad \phi_2 = \begin{Bmatrix} 0.47495 \\ -0.13017 \\ -0.53794 \end{Bmatrix}$$

$$\phi_3 = \begin{Bmatrix} 0.15111 \\ -0.50722 \\ 0.64572 \end{Bmatrix}$$

Application of Eq. (55), with the preceding modal information, yields the following response solutions:

$$U_1 = \frac{0.15792}{\omega_1^2 - \Omega^2} + \frac{-0.25550}{\omega_2^2 - \Omega^2} + \frac{0.09758}{\omega_3^2 - \Omega^2} \tag{56}$$

$$U_2 = \frac{0.25750}{\omega_1^2 - \Omega^2} + \frac{0.07002}{\omega_2^2 - \Omega^2} + \frac{-0.32752}{\omega_3^2 - \Omega^2} \tag{57}$$

$$U_3 = \frac{0.29366}{\omega_1^2 - \Omega^2} + \frac{0.28938}{\omega_2^2 - \Omega^2} + \frac{0.41696}{\omega_3^2 - \Omega^2} \tag{58}$$

From Eqs. (56–58), the existence of antiresonant frequencies and minimum response frequencies can be checked by application of the test criteria that are dependent on the product $(P_r P_{r+1})$, described earlier in this paper. The test conditions were applied to check for the existence of minimum response frequencies between each resonant frequency, for each coordinate response of the system. The results of the tests are presented in Table 1. Also in Table 1 are the antiresonant and minimum response frequencies computed for each coordinate response through use of the matrix formulations presented. As shown in Table 1, the indications obtained from the existence criteria that were applied to the direct response formulations were consistent with the computed frequency values obtained through application of the matrix formulations presented in this paper.

As previously stated, the matrix formulations presented allow for efficient determination of sensitivities of minimum response frequencies. For illustration, the following sensitivities of the U_2 minimum response frequencies were determined: $(\mathrm{d}\Omega_a/\mathrm{d}K_2)$, $(\mathrm{d}\Omega_m/\mathrm{d}K_2)$, $(\mathrm{d}\Omega_a/\mathrm{d}M_1)$, and $(\mathrm{d}\Omega_m/\mathrm{d}M_1)$. These sensitivity results were determined through application of the relations given by Eqs. (41) and (47). Also, the sensitivities were calculated through use of a central difference approach, with varied step sizes, for comparison to the matrix formulation results. For an arbitrary function, $f(x)$, the central difference method provides for approximation of the derivative as follows:

$$\left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=a} \cong \frac{f(a+\Delta x) - f(a-\Delta x)}{2(\Delta x)} \tag{59}$$

For each derivative quantity, three different step sizes were used so that an assessment of the validity of the matrix result could be established from the trends observed in the central difference results. The computed derivatives are presented in Tables 2 and 3. For all computed derivative quantities, as the step sizes in the central difference method are reduced, the computed sensitivities more closely approach the values computed from the matrix formulation sensitivity relationships. These results provide indication of the validity of the eigenvalue formulations for antiresonance and minimum response.

Certainly, the relatively flat spectral content in the vicinity of a minimum response point will present difficulty for accurate identification of the minimum response point frequency from experimental response data, particularly with the presence of measurement noise. However, the sensitivity of a minimum response point, to changes in mass or stiffness quantities, will include shifts of amplitude as well as shifts in the frequency. For illustration of this

Table 1 Comparison with existence criteria

			Matrix formulation	
Resp	$P_1 \cdot P_2$ [a]	$P_2 \cdot P_3$ [b]	Ω_a, rad/s	Ω_m, rad/s
U_1	−0.0403	−0.0249	-----	6.3606
				11.7184
U_2	+0.0180	−0.0229	8.1650	10.6283
U_3	+0.0850	+0.1207	6.2640	-----
			11.2885	

[a]Condition over range: $3.5092 < \Omega < 8.7065$ rad/s
[b]Condition over range: $8.7065 < \Omega < 13.3623$ rad/s

Table 2 Frequency sensitivity to stiffness change

ΔK_2, N/m	$\mathrm{d}\Omega_a/\mathrm{d}K_2$	$\mathrm{d}\Omega_m/\mathrm{d}K_2$
50.	0.0205764148	0.0166880976
5.0	0.0204140097	0.0170740448
0.5	0.0204124305	0.0170769682
Matrix formulation:	0.0204124145	0.0170769976

Table 3 Frequency sensitivity to mass change

ΔM_1	$\mathrm{d}\Omega_a/\mathrm{d}M_1$	$\mathrm{d}\Omega_m/\mathrm{d}M_1$
0.500	−1.38498245	−0.23574163
0.050	−1.36106394	−0.24490724
0.005	−1.36082999	−0.24496329
Matrix formulation:	−1.36082763	−0.24496385

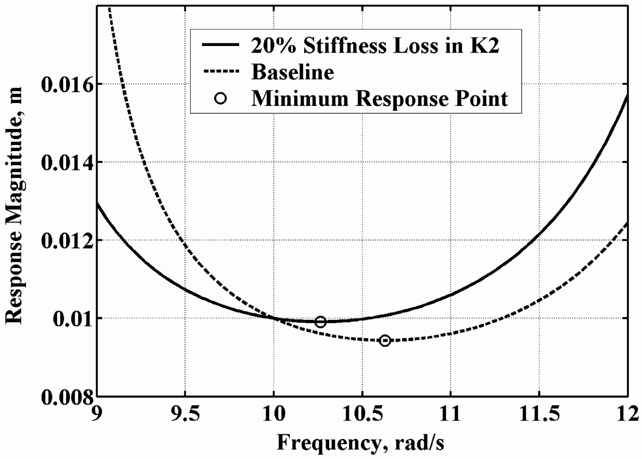

Fig. 4 Example shift in minimum response point induced by stiffness change.

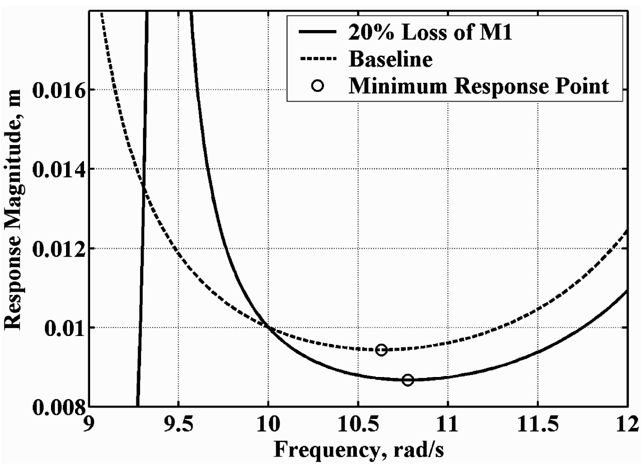

Fig. 5 Example shift in minimum response point induced by mass change.

effect, a 20% reduction of the stiffness of the spring element K_2 was applied. The resulting response characteristic is illustrated in Fig. 4 along with the corresponding response of the unmodified (baseline) system. Also, the response associated with a 20% reduction in the mass M_1 was determined for comparison to the baseline response and is shown in Fig. 5. The comparisons presented in Figs. 4 and 5 provide examples of not only the associated shifts of the minimum response points due to physical changes in the model, but also illustrate how the frequency responses in the vicinity of the minimum response points change as well.

Finally, it is important to note that a set of antisymmetric properties may be considered between minimum response points and resonance points. Changes in resonant frequencies are easily measured and, for that reason, are commonly used in practical applications. However, changes in response amplitudes at resonances may not be so easily measured depending on the sharpness of the resonance. For a given narrowband resolution in a frequency-domain response measurement, a sharper, or less damped resonance, will be defined by fewer points, which will result in less accuracy of the peak amplitude value. Conversely, although it is conceded that changes in a minimum response point frequency are inherently more difficult to monitor than corresponding changes in a resonant frequency, it is not unreasonable to expect that changes in the amplitude at a minimum response point may, in many situations, be more easily determined than corresponding changes in amplitude at resonance. Certainly, additional study is warranted for assessment of the feasibility for use of minimum response point information in a practical application.

Conclusions

Matrix formulations for determination of minimum responses for antiresonances and for local minima occurring at points with zero-slope in the frequency response were presented. A numerical example was also presented to illustrate use of the formulation and provide comparison to the traditional direct approach. It is clearly seen that the direct approach is only feasible for the simplest of problems. The orders of the polynomials involved in the direct representation of the response as a function of frequency quickly become difficult to handle with increasing size and complexity of the model. The matrix formulation provides a practical method for much larger models. In addition, with the formulation of the problem in terms of the system mass and stiffness matrices, it is possible to compute the sensitivities of the minimum response point frequencies directly from sensitivities of the system matrices. Thus, efficient reanalysis techniques can be used for applications for which changes in these minimum response frequencies need be determined.

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